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Nonequivalent periodic subsets of the lattice

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The use of Pólya's theorem in crystallography and other applications has greatly simplified many counting and coloring problems. Given a group of equivalences acting on a set, Pólya's theorem equates the number of unique subsets with the orbits of the group action. For a lattice and a given group of periodic equivalences, the number of nonequivalent subsets of the lattice can be solved using Pólya's counting on the group of relevant symmetries acting on the lattice. When equivalence is defined *via* a sublattice, the use of Pólya's theorem is equivalent to knowing the cycle index of the action of the group elements on a related finite group structure. A simple algebraic method is presented to determine the cycle index for a group element acting on a lattice subject to certain periodicity arguments.

1. Introduction

Using Pólya's theorem, we present a mathematical method to determine the number of nonequivalent periodic colorings of a lattice. This method is used to obtain explicit results in two dimensions, although the method generalizes to higher dimensions. There are a number of ways to define a lattice. We define a *lattice* to be a set of points generated as the integer combinations of some linearly independent basis. Periodicity on L is defined by a *sublattice* N of L, by requiring that for all $x \in L$, x and the coset $x + N$ have the same color; meaning that the coloring is periodic with respect to N. Two colorings of L are equivalent if there is a symmetry mapping one onto the other. Fig. 1 represents the integer lattice and the sublattice $4\mathbb{Z} \times 3\mathbb{Z}$. Fig. 2 demonstrates a periodic coloring of the lattice and an equivalent coloring obtained by translation.

Lattices which have a periodic subset such as in Fig. 2 are of interest to scientists studying crystallography and physical chemistry. J. S. Rutherford used Pólya's theorem and analytic number theory to enumerate transitionally equivalent periodic subsets of a lattice (Rutherford, 1992, 1993, 1995). Pólya's theorem has also been used in crystallographic applications classifying lattices by their symmetry components (Bernstein *et al.*, 1997). In later work, Rutherford enumerated unique subsets subject to rotation and reflection in two dimensions (Rutherford, 2009). Recent work in material science focuses on lattices subject to periodic boundary conditions and work by Forcade & Hart (2009) has provided an enumeration algorithm for nonequivalent periodic subsets given a lattice, sublattice and group of symmetries. In contrast, Pólya's theorem does not assist in enumerating unique subsets, but instead provides a formula (requiring polynomial expansion) to find the number of nonequivalent subsets.

We present a purely algebraic approach that can be used to determine the number of nonequivalent periodic colorings of a given lattice over the group of symmetries of the lattice. Since both translations and transformations (meaning reflections, rotations and introversion) are symmetries of a lattice, our work can be viewed as a generalization of Rutherford's results. The algebraic approach is based on solving simultaneous systems of congruences and can be extended to arbitrary dimension.

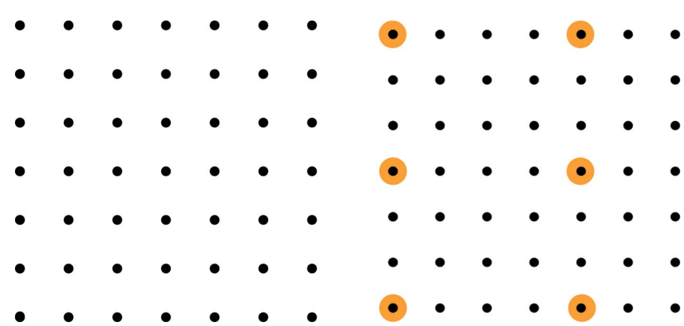

Figure 1
$\mathbb{Z}^2$ and the sublattice $4\mathbb{Z} \times 3\mathbb{Z}$.

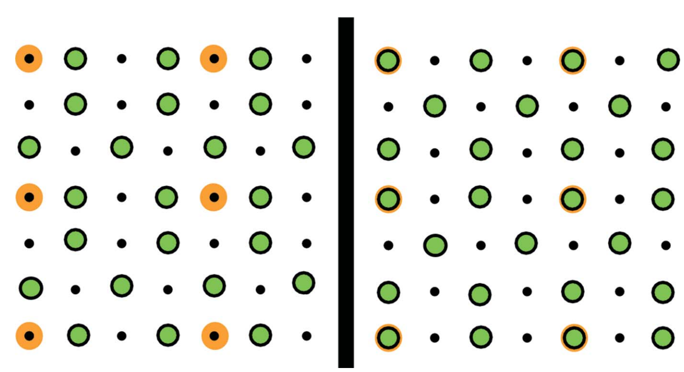

Figure 2
Translationally equivalent green coloring.

2. A brief mathematical explanation

We define periodicity on subsets of a lattice L, *via* a sublattice N, meaning one identifies the lattice with elements of the quotient group L/N. L/N can be represented by lattice points of L inside the unit cell of N, with addition defined *via* the periodicity. The lattice coloring is then defined by the coloring of L/N, which is a coloring of the lattice points of L in the unit cell of N, and then extended using periodicity. Using the fundamental theorem of finitely generated abelian groups, N can be defined to be $\{n_1 v_1, \ldots, n_k v_k\}$, for an appropriately chosen basis $\{v_1, \ldots, v_k\}$ of the base lattice L. Then $L/N \cong \mathbb{Z}^k/(n_1\mathbb{Z} \times \ldots \times n_k\mathbb{Z}) \cong \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_k}$.

We are naturally only interested in symmetries that preserve the lattice and sublattice. Once we identify the appropriate symmetries, we can then use Pólya's theorem on the action of the symmetries of the lattice on the quotient group L/N to count nonequivalent colorings. The use of Pólya's theorem is dependent on determining the cycle index of the action of a symmetry on the group (Read, 1987).

Viewing L/N as the lattice points of L inside the unit cell of N, with the periodicity defining the group operations, gives a natural geometric structure. The choice of basis for N will affect the unit cell of N and thus determine the action of the symmetries of L and N on the group L/N. Of course all the symmetries will act on L/N regardless of the shape of the unit cell. However, some unit cells will allow for a simpler calculation of the action than others. For example, a square unit cell can be deformed to a non-square parallelogram without changing the lattice or sublattice; both unit cells will admit the same symmetries. But, the action of 90° rotation is readily identified with the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

when acting on the square unit cell. One would have to apply the change-of-basis matrix to obtain the proper action for the parallelogram unit cell. We simplify calculations by restricting our discussion to symmetries which have the *standard action* on the unit cell, L and N, meaning the unit cell as a shape shares these symmetries, or in the case of 60° rotational symmetry the unit cell is the parallelogram formed by juxtaposing two equilateral triangles. All the explicit formulas given for the cycle index of a symmetry in two dimensions are given only for lattices whose components have these standard actions. Later in this paper we explain how one would modify the calculations to account for a change-of-basis formula to modify the action on the unit cell. We now establish a few facts about the group of symmetries of a lattice.

A symmetry can be represented *uniquely* as a transformation of L and N, followed by a translation of the lattice L. A symmetry could be identified by the underlying reflection or rotation of the lattice, and the translation of L using the semi-direct product. However, we have found that the following presentation simplifies calculations. For a symmetry g, we say $g = (\sigma, (m_1, \ldots, m_k))$, where σ is a transformation of L and $m = (m_1, \ldots, m_k)$ a translation. Meaning the action of g is to first perform the transformation σ, and then translate by m, we write

$$g = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1k} & m_1 \\ a_{21} & a_{22} & \ldots & a_{2k} & m_2 \\ & & \vdots & & \\ a_{k1} & a_{k2} & \ldots & a_{kk} & m_k \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix},$$

where a_{ij} is the jth coordinate of $\sigma(u_i)$ and $\{u_i\}$ represents the standard basis of the quotient group viewed as a k-dimensional vector space. For a given $x \in L/N$ we will define an element $x' \in L/N \times \{1\}$, such that

$$x' = \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ 1 \end{pmatrix}.$$

We will slightly abuse notation and refer to both x, x' as x, with the additional coordinate provided solely for computational purposes as demonstrated below. Then

$$g \cdot x = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1k} & m_1 \\ a_{21} & a_{22} & \ldots & a_{2k} & m_2 \\ & & \vdots & & \\ a_{k1} & a_{k2} & \ldots & a_{kk} & m_k \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ 1 \end{pmatrix}.$$

It is not difficult to show that multiplication of $g, g' \in G$ will result in another element of G. The other group axioms are also readily verified.

The transformation of a lattice in two or three dimensions has been studied and classified. For arbitrary dimension, the work by Kuzmanovich & Pavlichenkov (2002) establishes the finiteness of the symmetry group using techniques pioneered by Minkowski. For a lattice L and sublattice N, the group of symmetries is generated by the transformations that preserve both L and N, and translations of L that are invariant with respect to N.

3. Applying Pólya's theorem

Pólya's theorem can be used to solve coloring problems regarding the number of nonequivalent colorings of certain objects, where equivalence is determined *via* a group action. The cycle index of a group element g will be a polynomial of the form $\prod V_i^{\alpha_i}$, where the action of g on L/N viewed as an element of the symmetric group is the product of α_i i-cycles, α_{i+1} $(i+1)$-cycles *etc.* For a coloring consisting of k colors the polynomial is expanded using $V_i^{\alpha} = (x_1^i + \ldots + x_k^i)^{\alpha}$. The coefficient of $x_1^{j_1} \ldots x_k^{j_k}$ in the expanding polynomial will correspond to the number of nonequivalent colorings of j_1 elements with color x_1, j_2 elements with color x_2 *etc.* Read (1987) provides a thorough introduction to Pólya's theorem.

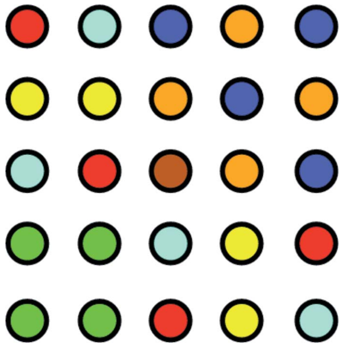

Figure 3
Orbits of the action of g on L/N.

4. The general method

The main step in determining the cycle index of a symmetry is to solve multiple systems of simultaneous congruences. To determine the cycle index of $g \in G$, we simply count the number of distinct solutions in L/N of $(g^r - I)x \equiv 0$ for a given r, where I is the standard $k+1$-dimensional identity matrix. This is equivalent to determining the number of $x \in L/N$ that satisfy $g^r x \equiv x_i \,(\bmod\, n_i)$. From group theory, we know that x is contained in an r-cycle iff $(g^r - I)x \equiv 0$ and $(g^t - I)x \not\equiv 0,\ t < r$.

Recall that two group elements x, y are said to be *conjugate* if there exists some group element g such that $gxg^{-1} = y$. For groups of matrices, two matrices are conjugate exactly when they are similar. Elements of the same conjugacy class will have the same cycle index (James & Liebeck, 2008). This means that symmetries consisting of conjugate transformations and identical translation elements will have the same conjugacy class. We shall analyze each case by conjugacy class of the transformation followed by an arbitrary translation. We will then sum over all different possible translation vectors to obtain the cycle index of all symmetries containing a given transformation. For convenience we define the *total cycle index* of a symmetry σ to be $\sum_{m \in L/N} \nu(g)$, where $g = (\sigma, m)$.

As mentioned above, it is possible that the basis for N will skew the unit cell and disguise symmetries. For a symmetry M_κ, we can diagonalize M_κ and find two matrices L, D such that $M_\kappa = LDL^{-1}$ where D is diagonal. Then we can compute powers of M_κ, by computing powers of LDL^{-1}. We then determine the cycle index polynomial using the same procedure: counting the number of solutions to $(M_\kappa^r - I)x \equiv 0$.

The next section contains a simple example that illustrates how to find the solutions to $(g^r - I)x \equiv 0$ in L/N.

5. A quick example

This example illustrates how we find the cycle index polynomial of a symmetry. Given the square base lattice $L = \mathbb{Z}^2$, the sublattice

$$N = 5\begin{pmatrix}1\\0\end{pmatrix} \times 5\begin{pmatrix}0\\1\end{pmatrix}$$

and the symmetry g consisting of 90° rotation followed by translation by the vector $(1, 0)$, we wish to determine the cycle index polynomial of g acting on L/N. We note that L/N is $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$, which is the set of integers tuples reduced modulo 5. We start by calculating $g^i \times x$ to get the following four systems of simultaneous congruences:

$$gx = \begin{pmatrix}0 & -1 & 1\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\1\end{pmatrix} = \begin{pmatrix}-x_2+1\\x_1\\1\end{pmatrix},$$

$$g^2x\begin{pmatrix}x_1\\x_2\\1\end{pmatrix} = \begin{pmatrix}-x_1+1\\-x_2+1\\1\end{pmatrix},\ g^3x = \begin{pmatrix}x_2\\-x_1+1\\1\end{pmatrix},$$

$$g^4x = \begin{pmatrix}x_1\\x_2\\1\end{pmatrix}.$$

Since g has order 4 we know that all elements are contained in either 1-, 2- or 4-cycles. For 1-cycles we have the following system of congruences:

$$\begin{aligned} x_1 &\equiv -x_2 + 1 \,(\bmod\, 5)\\ x_2 &\equiv x_1 \,(\bmod\, 5)\\ 1 &\equiv 1. \end{aligned}$$

Through substitution we see that $2x_1 \equiv 1\,(\bmod\, 5) \rightarrow x_1 \equiv 3$, $x_2 \equiv 3$ is the only possible 1-cycle. Solving for 2-cycles gives us

$$\begin{aligned} x_1 &\equiv -x_1 + 1 \,(\bmod\, 5)\\ x_2 &\equiv -x_2 + 1 \,(\bmod\, 5)\\ 1 &\equiv 1. \end{aligned}$$

We see that $(x_1, x_2) \equiv (3, 3)\,(\bmod\, 5)$ is the only possible 2-cycle. But this is actually our 1-cycle and there are consequently no 2-cycles. Then all other elements are in 4-cycles, meaning that g has a cycle index polynomial of

$$\underbrace{V_1^1}_{(3,3)}\ \underbrace{V_2^0}_{\text{no 2-cycles}}\ V_4^{\frac{24}{4}}.$$

Fig. 3 shows how g partitions L/N. Each color represents a different orbit, meaning that the action of g will permute the four yellow points among themselves. The center point is fixed by g and thus the only member of its cycle. There is a single fixed point and six 4-cycles.

If we use only two colors then

$$V_1^1 V_4^6 \sim (x+y)^1 (x^4 + y^4)^6.$$

We would need to sum over all the desired group elements in order to apply Pólya's theorem. The coefficient of $x^i y^j$ in the expansion would be the number of nonequivalent periodic colorings of L containing i elements of color x and j elements of color y.

6. Solving systems of simultaneous congruences

The key to determining the cycle index polynomial is to solve various systems of simultaneous congruences. The lemma below is an application of the Chinese remainder theorem with various moduli.

Lemma 6.1. $2x \equiv m \,(\mathrm{mod}\, n)$ has a unique solution for x when n is odd. When n is even, $\frac{1}{2}$ of the possible m will produce equations with two solutions for x, and the other possible m will produce inconsistent equations having no solutions for x.

Similarly

$$\begin{aligned} 2x_1 &\equiv m_1 \,(\mathrm{mod}\, n_1) \\ 2x_2 &\equiv m_2 \,(\mathrm{mod}\, n_2) \\ &\vdots \\ 2x_k &\equiv m_k \,(\mathrm{mod}\, n_k) \end{aligned}$$

has a unique solution for x when all the n_i are odd. Otherwise let $S = \{n_j : (2, n_j) = 2\}$, $h = |S|$. Then $(n^{k-h})/2^h$ of the possible $m = (m_1, \ldots, m_k)$ will each yield 2^h solutions for x. The other possible m will result in systems with no solutions.

For an abelian group A and a positive integer k, we use $\gamma(A, k)$ to denote the number of elements of order k in an abelian group A. Computing $\gamma(A, k)$ depends solely on the factorization of A using the fundamental theorem of finitely generated abelian groups. When $A = \mathbb{Z}/n\mathbb{Z}$, $\gamma(A, k) = \varphi(k)$ if $k|n$ and 0 otherwise; where φ is the Euler-totient function which measures the number of positive integers less than k and relatively prime to k.

7. Specific calculations in two dimensions

This section presents some examples of how to compute the total cycle index of a conjugacy class of symmetries in two dimensions. The method is the same for a general symmetry.

We will determine the total cycle index of elements conjugate to a 90° rotation. We then present a table of the cycle index polynomials for all two-dimensional transformations, where the transformations have the standard action on the lattice, sublattice and unit cell of the sublattice. The only possible symmetries of the lattice and sublattice in two dimensions are the standard symmetries 90, 180, 270°, reflection along either axis, or reflection along both axes, and rotation by 60, 120, 240, 300°. For a given $L/N \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$ we note that a given symmetry can only permute two coordinates, if the corresponding moduli are equal, *i.e.* $n_i = n_j$.

7.1. 90° rotation

The symmetry consisting of 90° rotation followed by a translation $m = (m_1, m_2)$ is represented by the following matrix:

$$M_{90} = \begin{pmatrix} 0 & -1 & m_1 \\ 1 & 0 & m_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Calculation reveals that $M_{90}^4 = I$, hence the n^2 elements of L/N are either in 1-, 2- or 4-cycles. We solve for 1-cycles first.

$$\begin{aligned} M_{90}x = x &\Longleftrightarrow \begin{pmatrix} 0 & -1 & m_1 \\ 1 & 0 & m_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} \\ &\Longleftrightarrow \begin{pmatrix} -x_2 + m_1 \\ x_1 + m_2 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}. \end{aligned}$$

Substitution gives us the equation

$$2x_1 = m_1 - m_2 \,(\mathrm{mod})\, n. \qquad (1)$$

From the lemma we see that solutions will vary depending on whether $n = n_1 = n_2$ is even or odd. First we assume n is even, then we note that $m_1 + m_2 \cong m_1 - m_2 \,(\mathrm{mod})\, 2$. Thus for $\frac{1}{2}$ of the possible $m \in L/N$ we will have two solutions for x_1 and each solution for x_1 will determine a unique solution for x_2; hence there are two solutions for (x_1, x_2). The other $\frac{1}{2}$ of the possible $m \in L/N$ will yield no solutions for (x_1, x_2), meaning there are no 1-cycles.

For n odd, the system gives a unique 1-cycle.

We now look for potential 2-cycles. We see that

$$\begin{aligned} M_{90}^2 x = x &\Longleftrightarrow \begin{pmatrix} -1 & 0 & m_1 - m_2 \\ 0 & -1 & m_1 + m_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -x_1 + m_1 - m_2 \\ -x_2 + m_1 + m_2 \\ 1 \end{pmatrix}. \end{aligned}$$

For n odd the system has a unique solution, that also satisfies equation (1); hence there is no 2-cycle. For n even we use the above lemma and find that there will be four possible solutions for (x_1, x_2) if $m_1 - m_2$ and $m_1 + m_2$ are both even. But $m_1 - m_1$ is even iff $m_1 + m_2$ is even. Two of these solutions will satisfy equation (1) above, so there are two elements contained in a single 2-cycle.

Since all other elements are in 4-cycles, total cycle indexes for rotation by 90° followed by arbitrary translation are

$$\sum_{m \in L/N} V_1^1 V_4^{\frac{n^2-1}{4}} = n^2 V_1^1 V_4^{\frac{n^2-1}{4}} \quad n \text{ odd}.$$

$$\sum_{\substack{m \in L/N \\ m_1 + m_2 \text{ even}}} V_1^2 V_2^1 V_4^{\frac{n^2-4}{4}} + \sum_{\substack{m \in L/N \\ m_1 + m_2 \text{ odd}}} V_4^{\frac{n_2}{4}} = \frac{n^2}{2} V_1^2 V_2^1 V_4^{\frac{n^2-4}{4}} + \frac{n^2}{2} V_4^{\frac{n^2}{4}} \quad n \text{ even}.$$

7.2. Table for two dimensions

Using the above techniques, Table 1 displays the symmetries for a generic $L/N \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$, where it is assumed that the transformations have the standard action on the unit cell.

Table 1
Two-dimensional symmetries.

Representative symmetry	Total cycle index
Identity	$\sum_{d\mid n_1 n_2} \gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, d) V_d^{\frac{n_1 n_2}{d}}$
180° rotation n_1, n_2 odd	$n_1 n_2 V_1^1 V_2^{\frac{n_1 n_2 - 1}{2}}$
180° rotation n_1 even, n_2 odd	$\frac{n_1 n_2}{2} V_1^2 V_2^{\frac{n_1 n_2 - 2}{2}} + \frac{n_1 n_2}{2} V_2^{\frac{n_1 n_2}{2}}$
180° rotation n_1, n_2 even	$\frac{n_1 n_2}{4} V_1^4 V_2^{\frac{n_1 n_2 - 4}{2}} + \frac{3 n_1 n_2}{4} V_2^{\frac{n_1 n_2}{2}}$
Reflection about an axis n_2 odd	$\sum_{d\mid n_1\,(d\,\text{odd})} n_2 \varphi(d) V_d^{\frac{n_1}{d}} V_{2d}^{\frac{n_1 n_2 - n_1}{2d}} + \sum_{d\mid n_1\,(d\,\text{even})} n_2 \varphi(d) V_d^{\frac{n_1 n_2}{d}}$
Reflection about an axis n_2 even	$\sum_{d\mid n_1\,(d\,\text{odd})} \frac{n_2 \varphi(d)}{2} \left(V_d^{\frac{2n_1}{d}} V_{2d}^{\frac{n_1 n_2 - 2n_1}{2d}} + V_{2d}^{\frac{n_1 n_2}{2d}} \right) + \sum_{d\mid n_1\,(d\,\text{even})} n_2 \varphi(d) V_d^{\frac{n_1 n_2}{d}}$
90° rotation n odd	$n^2 V_1^1 V_4^{\frac{n^2 - 1}{4}}$
90° rotation n even	$\frac{n^2}{2} V_1^2 V_2^1 V_4^{\frac{n^2 - 4}{4}} + \frac{n^2}{2} V_4^{\frac{n^2}{4}}$
Reflection about $x = y$ n odd	$\sum_{d\mid n} n \varphi(d) V_d^{\frac{n}{d}} V_{2d}^{\frac{n^2 - n}{2d}}$
Reflection about $x = y$ n even	$\sum_{d\mid n\,(d\,\text{odd})} n \varphi(d) V_d^{\frac{n}{d}} V_{2d}^{\frac{n^2 - n}{2d}} + \sum_{d\mid n\,(d\,\text{even})} n \varphi(d) V_{2d}^{\frac{n^2}{2d}}$
60° rotation $(n, 3) = 1$	$n^2 V_1 V_6^{\frac{n^2 - 1}{6}}$
60° rotation $(n, 3) = 3$	$\frac{n^2}{3} V_1 V_2^1 V_6^{\frac{n^2 - 3}{6}} + \frac{2n^2}{3} V_1 V_6^{\frac{n^2 - 1}{6}}$
120° rotation all n	$n^2 V_1 V_3^{\frac{n^2 - 1}{3}}$

8. An explicit example

In this section we present a sample calculation of the cycle index polynomial for the lattice $L = \mathbb{Z}^2$ and the sublattice generated by (0, 6) and (6, 0), $L/N = \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$. Both L and N are square and we will use the full group of symmetries, having $8 \times 36 = 288$ elements. The cycle index polynomial for this group will be

$$\underbrace{1V_1^6 + 3V_2^{18} + 8V_3^{12} + 24V_6^6}_{\text{the identity}} + \underbrace{36V_1^2 V_2^1 V_4^8 + 36V_4^9}_{90^\circ\text{ rotation}} + \underbrace{9V_1^4 V_2^{16} + 27V_2^{18}}_{180^\circ\text{ rotation}}$$
$$+ \underbrace{6V_1^{12} V_2^{12} + 6V_2^{18} + 12V_3^4 V_6^4 + 12V_6^6 + 12V_2^{18} + 24V_6^6}_{\text{reflection about an axis}}$$
$$+ \underbrace{12V_1^6 V_2^{15} + 24V_3^2 V_6^5 + 12V_4^9 + 24V_{12}^3}_{\text{reflection about } x=y}.$$

We expand using Pólya's theorem on two colors to get

$$x^{36} + x^{35}y + 9x^{34}y^2 + 40x^{33}y^3 + 282x^{32}y^4 + 1455x^{31}y^5 + 7278x^{30}y^6$$
$$+ 29849x^{29}y^7 + 107399x^{28}y^8 + 330369x^{27}y^9 + 890152x^{26}y^{10} + 2096153x^{25}y^{11}$$
$$+ 4364470x^{24}y^{12} + 8045195x^{23}y^{13} + 13215574x^{22}y^{14} + 19368689x^{21}y^{15}$$
$$+ 25423509x^{20}y^{16} + 29898089x^{19}y^{17} + 31566122x^{18}y^{18} + \ldots + y^{36}.$$

Note that we only need compute half of the coefficients since x, y are interchangeable. Looking at the coefficient of $x^{18}y^{18}$ tells us that there are 31566122 nonequivalent periodic colorings of our lattice with 18 elements colored one color and 18 elements colored another color. Equivalently we know that there are 31566122 nonequivalent periodic 18-element subsets of our lattice. Without equivalence the number of subsets is simply

$$\binom{36}{18} = 9705135300.$$

Taking equivalences has reduced the number of subsets to less than 1% of the original number.

9. Conclusion

As can be seen, the calculation of the cycle index for a given element is a simple procedure and the above techniques generalize completely to any arbitrary dimension. In higher dimensions there are additional symmetries to the lattice, many of which give more complicated systems of congruences. For example, when working with a symmetry component whose matrix is off-diagonal (meaning all the diagonal entries are zero), the system of simultaneous congruences will have solutions dependent not only on the factorization of the modulus n but also on the factorization of the dimension k.

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